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On Control Problem for Infinite System of Differential Equations of Second Order

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ABSTRACT

We study a control problem described by infinite system of differential equations of second order in the space l_{r+1}^2. Control parameter is subjected to integral constraint. Our goal is to transfer the state of the system from a given initial position to the origin for finite time. We obtained necessary and sufficient conditions on the initial positions for which the problem can be solved.

Keywords: infinite system, differential equation, control, integral constraint.

1. INTRODUCTION

Optimal control theory is an important and developing research area. Fundamental results are contained in many books (see, for example, Avdonin and Ivanov (1989), Butkovskiy (1975), Egorov (2004), Kirk (1998), Lee (1967), Pinch (1993), Pontryagin (19961), Pontryagin *et. al* (1969)). It is important because of its relevance in both theory and application and in other research areas such as economics and engineering. Specifically, control problems described by partial differential equations attract the attention of many researchers, for instance, Butkovskiy (1975), Chernous'ko (1992), Ibragimov (2002), Ibragimov (2005), Il'in (2001), Osipov (1977), Satimov and Tukhtasinov (2006), Satimov and Tukhtasinov (2007) and Satimov and Mamatov (2009) .

One of the approaches for solving such type of problems found in the literature is reduction the problem to the one described by infinite system of ordinary differential equations using the decomposition method (see, for example, Butkovskiy (1975) and Chernous'ko (1992)).

In Chernous'ko (1992) a control system described by the following linear partial differential equation

$$u_{tt} = Au + w, \qquad (1)$$

was reduced to the one described by the infinite system of ordinary differential equations

$$\ddot{z}_k(t) + \mu_k z_k(t) = w_k(t), \; k = 1, 2, \ldots, \qquad (2)$$

where in (1) $u = u(t, x)$ is a scalar function of $x \in R^n$ and time t; w is a control parameter,

$$Au = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right)$$

is a linear differential operator whose coefficients do not depend on t.

In (2) w_k, $k = 1, 2, \ldots,$ are control parameters and constants μ_k, $k = 1, 2, \ldots,$ satisfy

$$0 \le \mu_1 \le \mu_2 \le \to \infty$$

The reduction of the problem mentioned above suggests that it is possible to study the control problems described by infinite system of differential equations in one frame separately from the ones described by partial differential equations.

In this paper, we study control problem described by infinite system of second order equation (2) in the case of negative coefficients. The case of positive coefficients μ_k (as a differential game problem) was investigated in Il'in (2001) and Satimov and Tukhtasinov (2007). The results announced in these two papers are different from the one contained in this paper.

2. STATEMENT OF THE PROBLEM

Let $\lambda_1, \lambda_2, ...,$ be a bounded sequence of negative numbers and r be a real number. We introduce the space

$$l_r^2 = \{\alpha = (\alpha_1, \alpha_2, ...): \ \sum_{k=1}^{\infty} |\lambda_k|^r \ \alpha_k^2 < \infty\}$$

with inner product and norm

$$\langle \alpha, \beta \rangle_r = \sum_{k=1}^{\infty} |\lambda_k|^r \ \alpha_k \beta_k, \ \alpha, \ \beta \in l_r^2, \ \|\alpha\|_r = \left(\sum_{k=1}^{\infty} |\lambda_k|^r \ \alpha_k^2 \right)^{1/2}.$$

Let

$$L_2(t_0, T; l_r^2) = \left\{ w(t) = (w_1, w_2, ...) : \sum_{k=1}^{\infty} |\lambda_k|^r \int_{t_0}^{T} w_k^2(t) dt < \infty, \ w_k(\cdot) \in L_2(t_0, T) \right\}$$

$$\|w(\cdot)\|_{L_2(0, T; l_r^2)} = \left(\sum_{k=1}^{\infty} |\lambda_k|^r \int_{t_0}^{T} w_k^2(t) dt \right)^{1/2},$$

where $T, \ T > t_0,$ is a given number.

We examine the control problem described by the following infinite system of differential equations

$$\ddot{z}_k(t) + \lambda_k z(t) = w_k(t), \ z_k(0) = z_k^0, \ \dot{z}_k(0) = z_k^1, \ k = 1, 2, ..., \tag{3}$$

where
$z_k, w_k, \in R^1, \ k = 1, 2, ..., \ z_0 = (z_1^0, \ z_2^0, ...) \in l_{r+1}^2, \ z_1 = (z_1^1, \ z_2^1, ...) \in l_r^2,$
$w_1, w_2, ...,$ are control parameters.

Definition 1. A function $w(\cdot)$, $w : [0, T] \to l_r^2,$ with measurable coordinates $w_k(t), \quad 0 \le t \le T, \ k = 1, 2, ...,$ subject to

$$\sum_{k=1}^{\infty} |\lambda_k|^r \int_0^T w_k^2(s) ds \le \rho_0^2$$

where ρ is a given number, is referred to as the admissible control.

We denote the set of all admissible controls by $S(\rho)$.

Definition 2. A function $z(t) = (z_1(t), z_2(t), \ldots)$, $0 \le t \le T$, is called the solution of the equation (3) if each coordinate $z_k(t)$ of that

(1) is continuously differentiable on $(0, T)$, and satisfies the initial conditions

$$z_k(0) = z_k^0, \ \dot{z}_k(0) = z_k^1;$$

(2) has the second derivative $\ddot{z}_k(t)$ almost everywhere on $(0, T)$ that satisfies the equation

$$\ddot{z}_k(t) + \lambda_k z(t) = w_k(t), \quad k = 1, 2, \ldots,$$

almost everywhere on $(0, T)$.

The problems are to find conditions on initial positions such that $z(\tau) = 0$, $\dot{z}(\tau) = 0$ at some time τ, $0 \le \tau \le T$.

3. ANALYSIS OF THE PROBLEM

It is not difficult to verify that the kth equation in (3) has the unique solution

$$z_k(t) = z_k^0 \cosh(\alpha_k t) + z_k^1 \frac{\sinh(\alpha_k t)}{\alpha_k} + \int_0^t w_k(\tau) \frac{\sinh(\alpha_k (t-s))}{\alpha_k} ds, \tag{4}$$

where $\alpha_k = \sqrt{-\lambda_k}$. It's derivative is

$$\dot{z}_k(t) = \alpha_k z_k^0 \sinh(\alpha_k t) + z_k^1 \cosh(\alpha_k t) + \int_0^t w_k(\tau) \cosh(\alpha_k (t-s)) ds. \tag{5}$$

Let $C(t_0, T, l_r^2)$ be the space of continuous functions $z(t) = \left(z_1(t),\ z_2(t), \ldots\right)$, $0 \le t \le T$, with values in l_r^2. The following assertion is true (Avdonin and Ivanov (1989).

Assertion. If $\lambda_k < 0,\ k = 1, 2, \ldots,$ is a bounded below sequence, then the functions $z(t) = \left(z_1(t),\ z_2(t), \ldots\right)$, and $\dot{z}(t) = \left(\dot{z}_1(t),\ \dot{z}_2(t), \ldots\right)$ defined by (4) and (5) belong to the spaces $C(0,\ T;\ l_{r+1}^2)$ and $C(0,\ T;\ l_r^2)$, respectively.

Letting

$$\overline{x}_k(t) = \alpha_k z_k(t),\ \ x_{k0} = \alpha_k z_k^0,\ \ \overline{y}_k(t) = \dot{z}_k(t),\ \ y_{k0} = z_k^1$$

in (4) and (5), we obtain

$$\begin{cases} \overline{x}_k(t) = x_{k0}\cosh(\alpha_k t) + y_{k0}\sinh(\alpha_k t) + \int_0^t \sinh(\alpha_k(t-s))w_k(s)ds, \\ \overline{y}_k(t) = x_{k0}\sinh(\alpha_k t) + y_{k0}\cosh(\alpha_k t) + \int_0^t \cosh(\alpha_k(t-s))w_k(s)ds. \end{cases} \tag{6}$$

It is clear that $z_k(t) = 0,\ \dot{z}_k(t) = 0 \Leftrightarrow \overline{x}_k(t) = 0,\ \overline{y}_k(t) = 0.$ The latter equalities can be written as

$$A_k(t)\begin{bmatrix} x_{k0} \\ y_{k0} \end{bmatrix} + \int_0^t A_k(t)\begin{bmatrix} -\sinh(\alpha_k s) \\ \cosh(\alpha_k s) \end{bmatrix} w_k(s)ds = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

where

$$A_k(t) = \begin{bmatrix} \cosh(\alpha_k k) & \sinh(\alpha_k k) \\ \sinh(\alpha_k t) & \cosh(\alpha_k t) \end{bmatrix}.$$

Multiplying (7) by $A_k^{-1}(t)$, yields

$$\begin{cases} x_{k0} = \int_0^t \sinh(\alpha_k s)w_k(s)ds, \\ -y_{k0} = \int_0^t \cosh(\alpha_k s)w_k(s)ds. \end{cases} \tag{8}$$

Our task is now to find a control $w(\cdot)=(w_1(\cdot),w_2(\cdot),...)$ that satisfies (8) for all $k=1,2,....$ To this end we study the following set

$$X_k(\theta,\sigma)=\left\{(x_k,y_k):\ x_k=\int_0^\theta \sinh(\alpha_k s)w_k(s)ds,\right.$$
$$\left. y_k=\int_0^\theta \cosh(\alpha_k s)w_k(s)ds,\ w_k(\cdot)\in S(\sigma,\theta)\right\},$$

where

$$S(\sigma,\theta)\ =\ \left\{w_k(\cdot):\ \int_0^\theta w_k^2(s)ds\le\sigma^2, w_k(\cdot)\in L_2(0,\theta)\right\}.$$

It is clear that $(x_{k0},-y_{k0})\in X_k(\theta;\sigma)$ if and only if there exists $w_k(\cdot)\in S(\sigma,\theta)$ to satisfy (8).

We now study some properties of the set $X_k(\theta,\sigma)$. For short, we drop the indices *k*.

(i) *A control steering the state to the boundary of* $X_k(\theta,\sigma)$.

Let

$$z=\int_0^\theta(\sinh(\alpha s),\ \cosh(\alpha s))\,w(s)ds$$

be a point of $X_k(\theta,\sigma)$ and $e=(\xi,\eta)$ be a unit vector. Then by Cauchy-Schwarz inequality

$$\langle z,\ e\rangle=\int_0^\theta(\xi\sinh(\alpha s)+\eta\cosh(\alpha s))w(s)ds\le\sigma\gamma^{1/2}(\theta), \tag{9}$$

where

$$\gamma(\theta)=\int_0^\theta(\xi\sinh(\alpha s)+\eta\cosh(\alpha s))^2ds.$$

In (9) equality holds if

$$w(s)=\sigma\gamma^{-1/2}(\theta)(\xi\sinh(\alpha s)+\eta\cosh(\alpha s)) \tag{10}$$

almost everywhere on $[0,\theta]$.

It can be shown that if $w(\cdot)$ is defined by (10), then the point z is on the boundary $\partial X(\theta,\ \sigma)$ of the set $X(\theta,\ \sigma)$ and vice versa, if $z \in \partial X(\theta,\ \sigma)$, then $w(\cdot)$ has the form (10).

(ii) *A minimization problem.* We have

$$\gamma(\theta,\ e) = \xi^2 c_1 + 2\xi\eta c_2 + \eta^2 c_3,$$

where

$$c_1 = \frac{1}{4\alpha}\left(\sinh(2\alpha\theta) - 2\theta\right),\ c_2 = \frac{1}{4\alpha}\left(\cosh(2\alpha\theta) - 1\right),$$

$$c_3 = \frac{1}{4\alpha}\left(\sinh(2\alpha\theta) - 2\theta\right).$$

The solution to the extreme problem

$$\gamma(\theta,\ e) \to \min,\ \xi^2 + \eta^2 = 1,$$

is the minimum eigenvalue of the matrix

$$C = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix}.$$

One can verify that eigenvalues of C are

$$m_1 = \frac{1}{4\alpha}\left(\sinh(2\alpha\theta) + \sqrt{4\theta^2\alpha^2 + 4\sinh^4(\alpha\theta)}\right),$$

$$m_2 = \frac{1}{4\alpha}\left(\sinh(2\alpha\theta) - \sqrt{4\theta^2\alpha^2 + 4\sinh^4(\alpha\theta)}\right).$$

An easy computation shows that $m_1,\ m_2 > 0$ and

$$\lim_{\theta\to\infty} m_2 = \frac{1}{4\alpha}.$$

In addition, eigenvectors corresponding to m_1 and m_2 are $h_1 = \bar{h}_1 / |\bar{h}_1|$ and $h_2 = \bar{h}_2 / |\bar{h}_2|$ respectively, where

$$\bar{h}_1 = \left(\sinh^2(\alpha\theta),\ \sqrt{\alpha^2\theta^2 + \sinh^4(\alpha\theta)} + \alpha\theta\right),$$

$$\bar{h}_2 = \left(-\sinh^2(\alpha\theta),\ \sqrt{\alpha^2\theta^2 + \sinh^4(\alpha\theta)} - \alpha\theta\right).$$

Clearly, $h_1 \perp h_2$.

(iii) *The proof that* $\partial X(\theta, \sigma)$ *is an ellipse.*

Indeed, let $z \in \partial X(\theta, \sigma)$ and $h = \beta_1 h_1 + \beta_2 h_2$, $\beta_1^2 + \beta_2^2 = 1$. We assume that the numbers β_1, β_2 are chosen such that the control

$$w(t) = \frac{\sigma}{\sqrt{A(\theta, h)}}\left(h' \sinh(\alpha t) + h'' \cosh(\alpha t)\right),\ 0 \le t \le \theta,\ h = (h', h''),$$

steers the state $z(t)$ of the system

$$\begin{cases} \dot{z}_1 = \sinh(\alpha t) w(t), & z_1(0) = 0, \\ \dot{z}_2 = \cosh(\alpha t) w(t), & z_2(0) = 0, \end{cases} \tag{11}$$

from the point $z(0) = (0, 0)$ to the point $z(\theta)$, i.e.,

$$z = \left(\int_0^\theta \sinh(\alpha s) w(s) ds,\ \int_0^\theta \cosh(\alpha s) w(s) ds\right).$$

We have

$$\gamma(\theta) = c_1 (h')^2 + 2c_2 h' h'' + c_3 (h'')^2 = h^* C h = h^* \left(\beta_1 m_1 h_1 + \beta_2 m_2 h_2\right)$$

$$= \beta_1^2 m_1 + \beta_2^2 m_2. \tag{12}$$

Then

$$z=\int_0^{\theta}\begin{bmatrix}\sinh(\alpha s)\\ \cosh(\alpha s)\end{bmatrix}w(s)ds=\frac{\sigma}{\sqrt{\gamma(\theta,\ h)}}\int_0^{\theta}\begin{bmatrix}h'\sinh^2(\alpha s)+h''\sinh(\alpha s)\cosh(\alpha s)\\ h'\sinh(\alpha s)\cosh(\alpha s)+h''\cosh^2(\alpha s)\end{bmatrix}ds$$

$$=\frac{\sigma}{\sqrt{\gamma(\theta,\ h)}}\begin{bmatrix}c_1h'+c_2h''\\ c_2h'+c_3h''\end{bmatrix}=\frac{\sigma}{\sqrt{\gamma(\theta,\ h)}}Ch$$

$$=\frac{\sigma}{\sqrt{\gamma(\theta,\ h)}}(\beta_1m_1h_1+\beta_2m_2h_2).$$

If we denote $z_i=\frac{\sigma}{\sqrt{\gamma(\theta,\ h)}}\beta_i m_i,\ i=1,2,$ then in view of (12) we obtain

$$\left(\frac{z_1}{\sigma\sqrt{m_1}}\right)^2+\left(\frac{z_2}{\sigma\sqrt{m_2}}\right)^2=\frac{\beta_1^2m_1+\beta_2^2m_2}{\gamma(\theta,\ h)}=1.$$

Thus, $\partial X(\theta,\sigma)$ is an ellipse. Moreover, $\sigma\sqrt{m_2}$ is half length of its minor axis.

(iv) *If* $0\le\theta_1\le\theta_2$, *then* $X(\theta_1,\ \sigma)\subset X(\theta_2,\ \sigma)$.

Indeed, let $(x,\ y)\in X(\theta_1,\ \sigma)$. Then by the definition of $X(\theta,\ \sigma)$ there is an admissible control $w(t)\in S(\sigma,\theta_1)$ such that

$$x=\int_0^{\theta_1}\sinh(\alpha s)w(s)ds,\ \ y=\int_0^{\theta_1}\cosh(\alpha s)w(s)ds.$$

We now define another control

$$\overline{w}(t)=\begin{cases}w(t), & 0\le t\le\theta_1,\\ 0, & \theta_1<t\le\theta_2.\end{cases}$$

Observe $\overline{w}(t)\in S(\sigma,\theta_2)$ and

$$x=\int_0^{\theta_1}\sinh(\alpha s)w(s)ds=\int_0^{\theta_2}\sinh(\alpha s)\overline{w}(s)ds,$$

$$y=\int_0^{\theta_1}\cosh(\alpha s)w(s)ds=\int_0^{\theta_2}\cosh(\alpha s)\overline{w}(s)ds.$$

This means that $(x,\ y)\in X(\theta_2,\ \sigma)$. Therefore, $X(\theta_1,\ \sigma)\subset X(\theta_2,\ \sigma)$.

4. MAIN RESULT

a) Sufficient Condition

We now study a control problem of steering the state of the system (6) from $(x_{10}, y_{10},\ x_{20}, y_{20},...)$ to $(0,0,...)$.

Let

$$X(\theta,\ \sigma)=\bigcup_{(\sigma_1,\sigma_2,...),}\ \prod_{k=1}^{\infty}X_k(\theta,\ \sigma_k),$$

where $\prod$ is Cartesian product of the sets $X_k(\theta,\ \sigma_k)$ and the union is taken over all the sequences $\sigma_1,\ \sigma_2,\cdots$ such that

$$\sum_{k=1}^{\infty}\sigma_k^2=\sigma^2,\ \sigma_k\ge 0.$$

Let $\tilde{z}_0=((x_{10},\ -y_{10}),\ (x_{20},\ -y_{20}),...)$. The following theorem is true.

Theorem 1. *If* $\tilde{z}_0\in X(\theta,\ \sigma)$, *then there exists a control* $w(\cdot)\in S(\sigma,\theta)$ *such that* $z_k(\theta)=\dot{z}_k(\theta)=0,\ k=1,2,...,$ *in (4) and (5).*

Proof

Let $\tilde{z}_0\in X(\theta,\ \sigma)$, then by definition of $X(\theta,\ \sigma)$ there exists a sequence $\sigma_1,\sigma_2,...,\ \sum_{k=1}^{\infty}\sigma_k^2=\sigma^2,\ \sigma_k\ge 0,\ k=1,2,...,$ such that

$$((x_{10},\ -y_{10}),\ (x_{20},\ -y_{20}),...)\in\prod_{k=1}^{\infty}X_k(\theta,\ \sigma_k).$$

Consequently,

$$(x_{k0},\ -y_{k0})\in X_k(\theta,\ \sigma_k),\ k=1,2,\cdots.$$

Then by definition of $X_k(\theta,\ \sigma_k)$ there exists

$$w_{0k}(\cdot)\in S(\sigma_k,\theta),\ k=1,2,...,$$

such that

$$x_{k0} = \int_0^\theta \sinh(\alpha_k s) w_{0k}(s) ds,$$
$$-y_{k0} = \int_0^\theta \cosh(\alpha_k s) w_{0k}(s) ds.$$

These mean $\bar{x}_k(\theta) = 0,\ \bar{y}_k(\theta) = 0,\ k = 1, 2, \ldots$. Therefore, $z(\theta) = \dot{z}(\theta) = 0.$

The proof of the theorem is complete.

b) Necessary Condition

Let find the limit set of the set $X_k(\theta, \sigma_k)$. Since the boundary of the set $X_k(\theta, \sigma_k)$ is an ellipse, the shortest distance of its center from the boundary is $\sigma_k \sqrt{m_2(\theta)}$, and the longest one is $\sigma_k \sqrt{m_1(\theta)}$. As

$$\lim_{\theta \to \infty} \sigma_k \sqrt{m_2(\theta)} = \frac{\sigma_k}{2\sqrt{\alpha_k}},\ \lim_{\theta \to \infty} \sigma_k \sqrt{m_1(\theta)} = \infty,$$
$$h_{10} = \lim_{\theta \to \infty} h_2 = \frac{1}{\sqrt{2}}(-1, 1),\ h_{20} = \lim_{\theta \to \infty} h_2 = \frac{1}{\sqrt{2}}(1, 1),$$

then the limit set $X_k(\sigma_k)$ for $X_k(\theta, \sigma_k)$ as $\theta \to \infty$ is

$$X_k(\sigma_k) = \left\{ z = ph_{10} + qh_{20} \ : \ p \in \left(-\frac{\sigma_k}{2\sqrt{\alpha_k}}, \frac{\sigma_k}{2\sqrt{\alpha_k}} \right),\ q \in R \right\}$$
$$= \left\{ (x,\ y) :\ \sqrt{2}\, | x - y | \sqrt{\alpha_k} < \sigma_k \right\}.$$

The following theorem is true.

Theorem 2. *If*

$$2 \sum_{k=1}^{\infty} \alpha_k \, | x_{k0} - y_{k0} |^2 \geq \sigma^2, \tag{13}$$

then the point $((x_{10},\ y_{10}), (x_{20},\ y_{20}), \ldots)$ *cannot be steered to the origin.*

Proof

Since $X_k(\theta_1, \sigma_k) \subset X_k(\theta_2, \sigma_k)$, $(\theta_1 < \theta_2)$, $X_k(\theta, \sigma_k) \subset X_k(\sigma_k)$, therefore, if we show that $(x_{m0}, y_{m0}) \notin X_m(\sigma_m)$ for some m, then we shall have established the theorem.

Suppose the assertion of the theorem is false. Then there exists a control

$$w(t) = (w_1(t), w_2(t), \ldots),\ 0 \le t \le T,\ w(\cdot) \in S(\sigma, T),$$

such that

$$x_{k0} = \int_0^T \sinh(\alpha_k t) w_k(t) dt,$$
$$y_{k0} = \int_0^T \cosh(\alpha_k t) w_k(t) dt.$$

i.e., $(x_{k0},\ y_{k0}) \in X_k(T,\ \sigma_k)$, where $\int_0^T w_k^2(t)dt = \sigma_k^2$. Then (13) implies that there is m such that

$$2\alpha_m \,|\, x_{m0} - y_{m0} \,|^2 \ge \sigma_m^2,$$

and so $(x_{m0},\ y_{m0}) \notin X_m(\sigma_m)$. Hence, $(x_{m0},\ y_{m0}) \notin X_m(T, \sigma_m)$.

This is a contradiction. Therefore, the proof of the theorem is complete.

5. CONCLUSION

In this paper, we have examined control problems described by the infinite system of differential equations (3) when the coefficients λ_k are negative numbers. We have proved two theorems. The first one is a sufficient condition. Here, we have constructed a set $X(\theta, \sigma)$ and showed that if $\tilde{z}_0$ belongs to this set then the point $(z(\theta), \dot{z}_k(\theta))$ can be steered to $(0,0)$. The second theorem is a necessary condition. According to this theorem if the point $((x_{10},\ y_{10}), (x_{20},\ y_{20}), \ldots)$ can be steered to the origin then it must satisfy the condition $2\sum_{k=1}^{\infty} \alpha_k \,|\, x_{k0} - y_{k0} \,|^2 < \sigma^2$.

The results of the present paper can be developed in two directions: 1) in obtaining necessary and sufficient condition of steering the point $((x_{10},\ y_{10}),(x_{20},\ y_{20}),...)$ to the origin; 2) in studying differential game problems for such systems.

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